Limits

(Draft)

Resource Supplement for

Pre-Calculus B 120

The NB Department of Education and Early Childhood Development
gratefully acknowledges the work done by the *Limits Resource Development Committee* of
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to produce this resource for use by New Brunswick Teachers.

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Introduction

This resource has been produced by New Brunswick teachers to supplement the core text resource, *Pre-Calculus 12* (MHR 2012) in which the topic of limits is not covered.

The concept of a limit is essential to the development and understanding of Calculus. Limits are used in the definition of the continuity of a function and are used to develop the concepts of instantaneous rates of change (slope of a tangent, derivative) and the area under a curve (integrals). These concepts have many applications in Science, Social Science, and Business.

The ideas that would develop into Calculus began with the Ancient Greeks. Zeno developed several paradoxes that could not be explained at the time. One of these involved reaching a destination by going half the remaining distance in each step. He said that you would never be able to reach the destination because you would always have half the remaining distance to travel. Archimedes explored a similar idea, using what was called the process of exhaustion to find the area of a circle. He placed regular polygons with increasing number of sides inside a circle and determined their areas. As the number of sides increased the area of the polygon approached the area of the circle. Although both of these mathematicians were talking about the concept we now refer to as limits, the definition was not formalized until the early 1800's. Many scholars, including Cavalieri, Fermat, Descartes, Barrow, Newton and Leibnitz developed the concepts of Calculus in the intervening years but the formal definition of a limit was not given until Cauchy (1789 – 1857) used limits to define continuity. The following websites have more on the history of calculus:
http://www.uiowa.edu/~c22m025c/history.html
http://www.mscs.dal.ca/~kgardner/History.html , http://faculty.unlv.edu/bellomo/Math714/Notes/11_Calculus.pdf
http://www.andrewsaladino.com/calculus/history.html .

Students may have previously heard the term 'limit' in relation to horizontal asymptotes (exponential, reciprocal, and rational functions) and vertical asymptotes (logarithmic, reciprocal, and rational functions). The ideas would also have been discussed in conjunction with infinite sequences and series. Students will need to be shown the notation,

$$\lim_{x \to a} f(x) = L$$

and how it is used to describe asymptotes, continuity, sequences and series. The approach in the four limit outcomes in *Pre-Calculus B 120* is in more depth than that in previous courses. In this course, students will not only evaluate limits but will also apply their understanding of these concepts to better understand the behavior of functions. Specifically, limits will be used to describe how a function behaves near a specific point but not at that point. The value of the limit does not depend on the value of the function. Limits will also be used to help determine whether or not a function is continuous and determine its end behavior.

Please note that as students begin Calculus they will see set builder notation and interval notation used interchangeably. They should be allowed to use either type of notation in their solutions. In Interval Notation, to indicate that one of the endpoints is to be excluded from the set, the corresponding square bracket can be replaced with a parenthesis (in another convention, instead of the parenthesis, the square bracket is reversed). Examples of both notations are shown below. Real numbers are assumed unless a different set of numbers is stated.

Set Builder Notation	Interval Notation
$\{x \mid -2 \le x \le 4\}$	$x \in [-2,4]$
$\{x \mid -2 < x \le 4\}$	$x \in (-2,4]$
$\{x \mid -2 \le x < 4\}$	$x \in [-2,4)$
$\{x \mid -2 < x < 4\}$	$x \in (-2,4)$
$\{x \mid x \le 4\}$	$x \in (-\infty, 4]$
$\{x \mid x < 4\}$	$x \in (-\infty, 4)$
$\{x \mid x \ge 4\}$	$x \in [4, \infty)$
$\{x \mid x > 4\}$	$x \in (4, \infty)$

L1. Determine the limit of a function at a point both graphically and analytically.

Achievement Indicators

- Demonstrate an understanding of the concept of limit at a point, and the notation to express the limit of a function $f(x)$ as x approaches a as $\lim_{x\to a} f(x) = L$
- Evaluate the limit of a function at a point graphically, analytically and using tables of values.
- Distinguish between the limit of a function as x approaches a and the y-value of the function at $x = a$.
- Use the properties of limits to evaluate a limit by substitution of the x-value into $f(x)$, where possible.
- Explore undefined limits (limits that do not exist (DNE)).
- Explore limits that arise, when substituting yields an indeterminate form.

Limits describe how functions behave as the independent variable approaches a given value. For example, the graph and **Table 1** for the function $f(x) = x^2 - x + 2$, are shown below. The values of $f(x)$ get closer and closer to 4 as x gets closer and closer to 2.

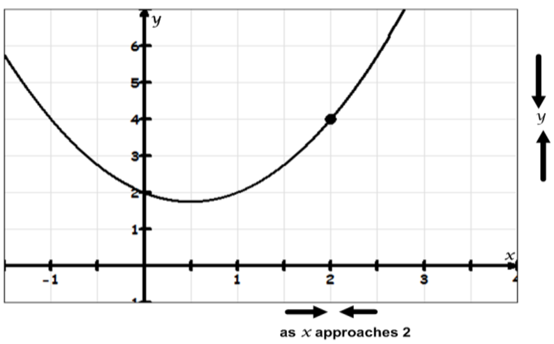


Table 1

Approaching 2 from the left

$x < 2$	$f(x)$
1.0	2
1.5	2.75
1.9	3.71
1.99	3.9701
1.999	3.997001

Approaching 2 from the right

$x > 2$	$f(x)$
3.0	8
2.5	5.75
2.1	4.31
2.01	4.0301
2.0001	4.003001

Properties of Limits

If the following limits both exist

$\lim_{x\to a} f(x) \qquad \lim_{x\to a} g(x)$

and c is a constant, then the following properties hold:

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[c\, f(x)] = c \lim_{x\to a} f(x)$$

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \; \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad if \; \lim_{x\to a} g(x) \neq 0$$

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n \; if\ n\ is\ a\ positive\ integer$$

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} \quad if\ the\ root\ on\ the\ right\ side\ exists$$

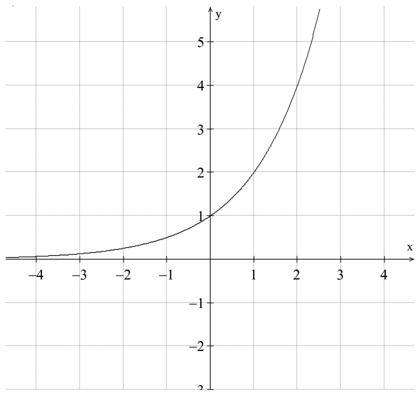
L1: Determine the limit of a function at a point both graphically and analytically.

L1 Exercises

1. Find the limit of each function shown.

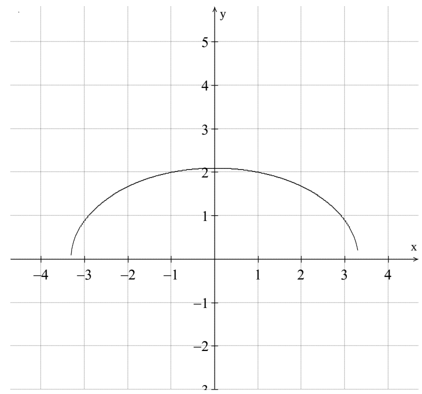
a)

$\lim_{x \to 1} f(x)$



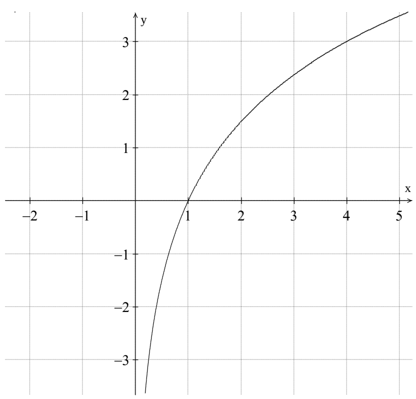
b)

$\lim_{x \to -1} f(x)$



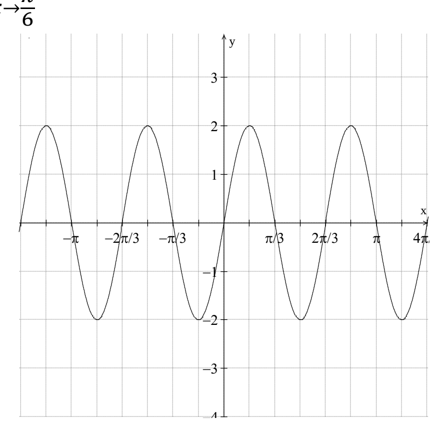
c)

$\lim_{x \to 1} f(x)$



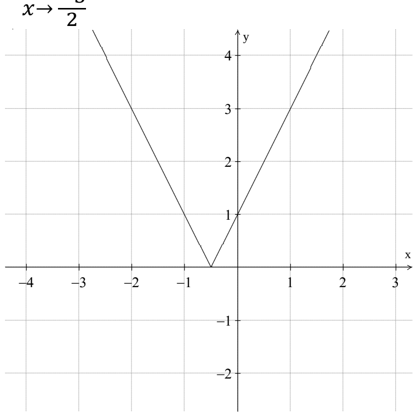
d)

$\lim_{x \to \frac{\pi}{6}} f(x)$



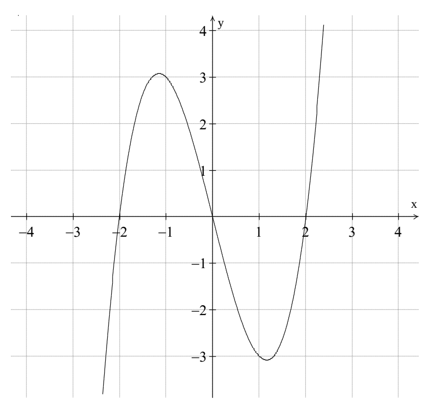
e)

$\lim_{x \to \frac{-3}{2}} f(x)$



f)

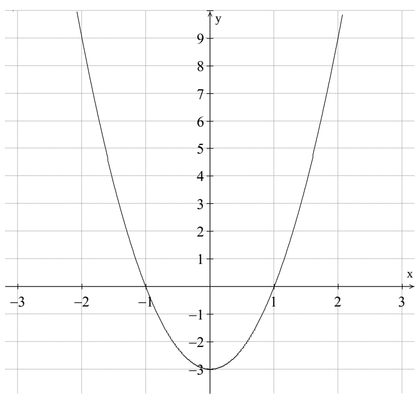
$\lim_{x \to -1} f(x)$



L1: Determine the limit of a function at a point both graphically and analytically.

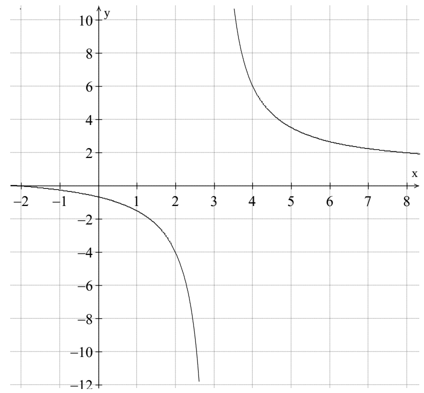
g)

$$\lim_{x \to 2} f(x)$$



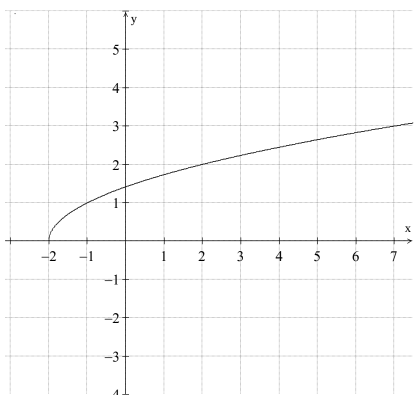
h)

$$\lim_{x \to 4} f(x)$$



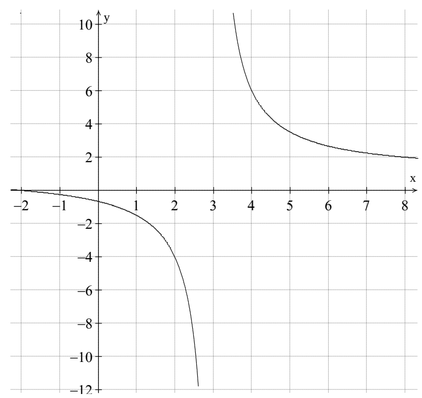
i)

$$\lim_{x \to 2} f(x)$$



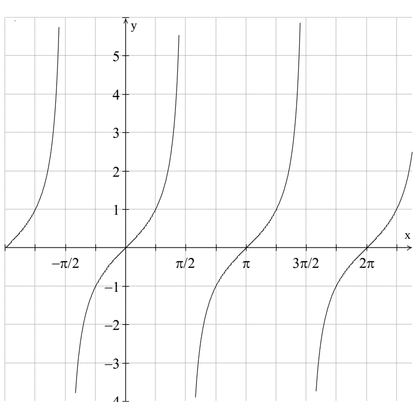
j)

$$\lim_{x \to 3} f(x)$$



k)

$$\lim_{x \to \pi/2} f(x)$$



l)

$$\lim_{x \to 3} f(x)$$

x	$f(x)$
2	-1
2.5	-3
2.9	-19
2.99	-199
2.999	-1999
3	
3.001	2001
3.01	201
3.1	21
3.5	5
4	3

L1: Determine the limit of a function at a point both graphically and analytically.

2. Evaluate the following limits:

a) $\lim_{x \to 4} \dfrac{x-4}{x^3-64}$

b) $\lim_{x \to 7} \dfrac{\sqrt[5]{3-5x}}{(x-5)^3}$

c) $\lim_{x \to -3} \dfrac{x+3}{3-\sqrt{x+12}}$

d) $\lim_{x \to -3} \sqrt[3]{\dfrac{x-4}{6x^2+2}}$

e) $\lim_{x \to 3} \dfrac{\frac{1}{x^2}-\frac{1}{9}}{x-3}$

f) $\lim_{h \to 0} \dfrac{(-3+h)^2-9}{h}$

g) $\lim_{h \to 0} \dfrac{(2+h)^3-8}{h}$

h) $\lim_{x \to 0} \dfrac{\frac{1}{1+x}-1}{x}$

i) $\lim_{x \to -1} \dfrac{2x+3}{3x+2}$

j) $\lim_{x \to -8} \dfrac{x+8}{\sqrt[3]{x}+2}$

k) $\lim_{x \to -1} \dfrac{8}{(3+x)\sqrt{3-x}}$

l) $\lim_{x \to 9} \dfrac{x-9}{3-\sqrt{x}}$

m) $\lim_{x \to 0} \dfrac{(3+x)^3-3^3}{(3+x)^2-3^2}$

n) $\lim_{x \to 9} \dfrac{x^2-81}{3-\sqrt{x}}$

o) $\lim_{x \to 16} \dfrac{2\sqrt{x}+x^{3/2}}{\sqrt[4]{x}+5}$

p) $\lim_{x \to 2} \dfrac{x^2-4}{3x^2-7x+2}$

q) $\lim_{x \to 7} \dfrac{\sqrt{x+2}-3}{x-7}$

r) $\lim_{x \to 4} \left(\dfrac{8}{x^2-16} - \dfrac{1}{x-4}\right)$

s) $\lim_{x \to 5} \dfrac{2x}{x-5}$

t) $\lim_{x \to 0} \dfrac{x^3}{2x^3+3x^4}$

u) $\lim_{x \to 2} \dfrac{x^2-2x}{2x^2-7x+6}$

v) $\lim_{x \to \pi} \dfrac{1+\sin x}{\cos x}$

w) $\lim_{x \to \frac{\pi}{4}} (\cos x) + (\sin x)$

x) $\lim_{x \to \frac{3\pi}{2}} \dfrac{\sin x}{\cos x}$

L1 Solutions

1. a) 2 **b)** ≈ 2 **c)** 0 **d)** 2 **e)** 2 **f)** 3 **g)** 9 **h)** 6 **i)** 2 **j)** DNE **k)** DNE **l)** DNE

2. a) $\dfrac{1}{48}$ **b)** $-\dfrac{1}{4}$ **c)** -6 **d)** $-\dfrac{1}{2}$ **e)** $-\dfrac{2}{27}$ **f)** -6 **g)** 12 **h)** -1

i) -1 **j)** 12 **k)** 2 **l)** -6 **m)** $\dfrac{9}{2}$ **n)** -108 **o)** $\dfrac{72}{7}$ **p)** $\dfrac{4}{5}$ **q)** $\dfrac{1}{6}$

r) $-\dfrac{1}{8}$ **s)** DNE **t)** $\dfrac{1}{2}$ **u)** 2 **v)** -1 **w)** $\sqrt{2}$ **x)** DNE

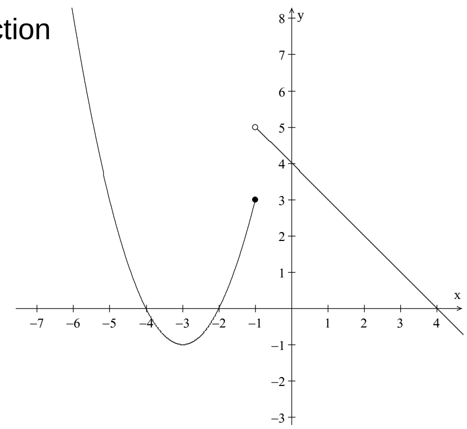
L2. Explore one-sided limits graphically and analytically.

Achievement Indicators

- Explore and analyze left and right hand limits as x approaches a certain value using correct notation.
- Evaluate the left and right hand limits of piecewise functions.

Example 1

Consider the following piecewise defined function as, $f(x) = \begin{cases} x^2 + 6x + 8 & x \leq -1 \\ -x + 4 & x > -1 \end{cases}$



What y value is the function approaching as x approaches -1 from the left?

Find: $\lim_{x \to -1^-} f(x)$

(Note that the '$-$' superscript as an exponent on -1, means 'from the left side'.)

Substituting the x value of -1 into the function $f(x) = x^2 + 6x + 8$ (*since -1 is in the domain of this function*), gives a y value of 3.

$\therefore \lim_{x \to -1^-} f(x) = 3$

What y-value is the function approaching as x approaches -1 from the right?

Find: $\lim_{x \to -1^+} f(x)$

(Note that the '$+$' superscript as an exponent on -1, means 'from the right side'.)

The x value of -1 cannot be substituted into the function $f(x) = -x + 4$ because -1 is not in the domain of this function. **Table 2**, shows that x gets closer and closer to -1 from the right side, the y-values get closer and closer to 5. Since limits are used to determine the value a function **approaches** a value, we can conclude that the limit is 5.

Table 2

Approaching -1 from the right

$x > -1$	$f(x)$
1.0	3
0.5	3.5
0	4
-0.05	4.5
-0.09	4.9

$\therefore \lim_{x \to -1^+} f(x) = 5$

L2: Explore one-sided limits graphically and analytically.

For each case, students can trace a finger along the graph (from each side) to see what y value the function approaches as the x value is approached. It is important to highlight that even though a function does not exist at a particular x value (empty circle), there is still a limit as shown in the example.

To find the limit for the piecewise function defined in the example, the limit for each piece has been determined as:

$\lim_{x \to -1^-} f(x) = 3$

$\lim_{x \to -1^+} f(x) = 5$

Since the limit from the left does not equal the limit from the right, then the general limit for this piecewise function does not exist ($page$ 51 of the *Pre-Calculus B 120* curriculum document).

Example 2

Given $f(x)\begin{cases} -x^2 - 4x - 2 & x \le 0 \\ (x-1)^3 - 1 & x > 0 \end{cases}$

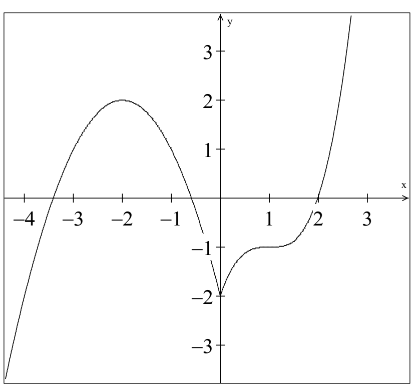
a) Sketch the graph of $f(x)$.

b) Find the following limits, if they exist.

i) $\lim_{x \to 0^-} f(x)$ ii) $\lim_{x \to 0^+} f(x)$ iii) $\lim_{x \to 0} f(x)$

Solution:

a)



b) i) $\lim_{x \to 0^-} f(x) = -2$

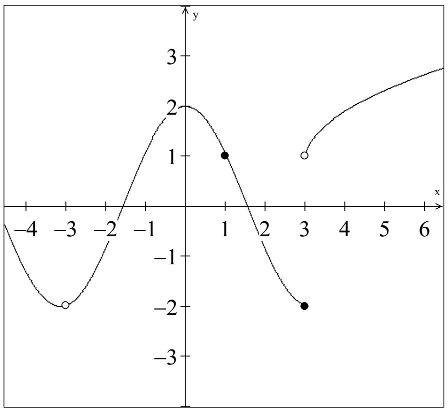
ii) $\lim_{x \to 0^+} f(x) = -2$

iii) Since the limit from the left and from the right are equal then, $\lim_{x \to 0} f(x) = -2$

L2: Explore one-sided limits graphically and analytically.

L2 Exercises

1. Given the graph of $f(x)$, find the following limits.



a) $\lim_{x\to -3^-} f(x)$ b) $\lim_{x\to -3^+} f(x)$ c) $\lim_{x\to -3} f(x)$

d) $\lim_{x\to 1^+} f(x)$ e) $\lim_{x\to 1^-} f(x)$ f) $\lim_{x\to 1} f(x)$

g) $\lim_{x\to 3^-} f(x)$ h) $\lim_{x\to 3^+} f(x)$ i) $\lim_{x\to 3} f(x)$

2. Sketch each piecewise function below and find the limit at the indicated point. If the limit does not exist, provide an explanation.

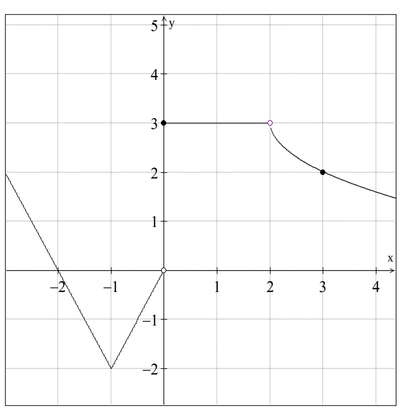
a) $f(x) = \begin{cases} 2 & x < 1 \\ 3 & x = 1 \\ x+1 & x > 1 \end{cases}$ Find $\lim_{x\to 1} f(x)$

b) $f(x) = \begin{cases} 4 - x^2 & -2 < x \le 2 \\ x - 2 & x > 2 \end{cases}$ Find $\lim_{x\to 2} f(x)$

c) $f(x) = \begin{cases} |x+2| + 1 & x < -1 \\ -x + 1 & -1 \le x \le 1 \\ x^2 - 2x + 2 & x > 1 \end{cases}$ Find $\lim_{x\to 1} f(x)$

3. a) What is the possible defining function for the piecewise graph below?

b) i) Does the limit exist at $x = 0$?
ii) Does the limit exist at $x = 2$?



4. The function $f(t)$ is defined by $f(t) = \begin{cases} 3t + b & t < 1 \\ 2 - bt^2 & t \ge 1 \end{cases}$ where b is a constant.

Compute $\lim_{t\to 1^+} f(t)$ *and* $\lim_{t\to 1^-} f(t)$ in terms of b.

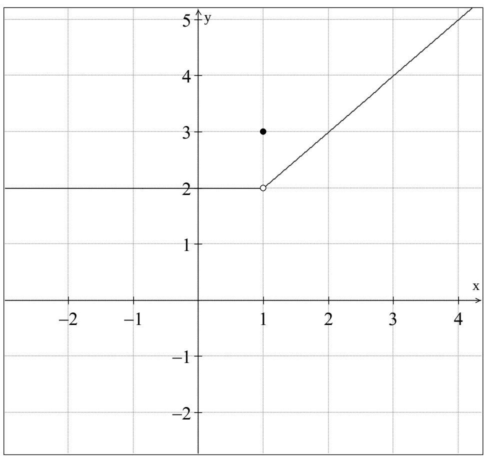
L2 Solutions

1. a) $\lim_{x\to -3^-} f(x) = -2$ b) $\lim_{x\to -3^+} f(x) = -2$ c) $\lim_{x\to -3} f(x) = -2$

d) $\lim_{x\to 1^+} f(x) = 1$ e) $\lim_{x\to 1^-} f(x) = 1$ f) $\lim_{x\to 1} f(x) = 1$

g) $\lim_{x\to 3^-} f(x) = -2$ h) $\lim_{x\to 3^+} f(x) = 1$ i) $\lim_{x\to 3} f(x)$ *does not exist*

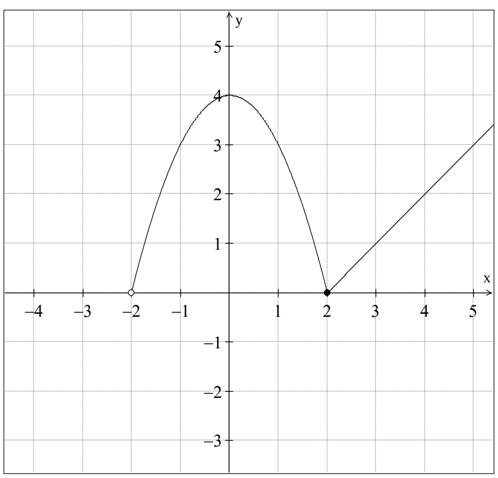
2. a)



$\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 2$

Since the limits from the left and right of 1 are equal, $\lim_{x\to 1} f(x) = 2$

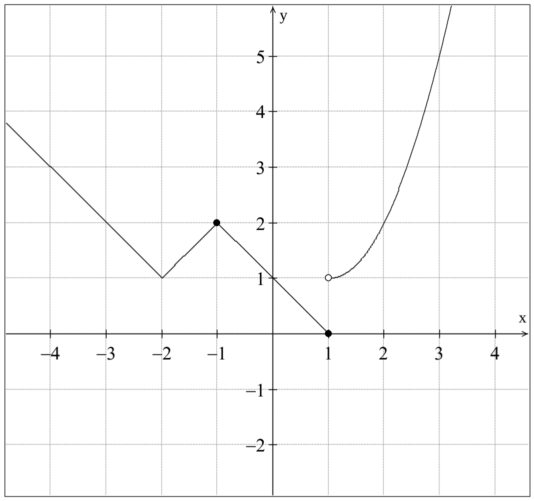
b)



$\lim_{x\to 2^-} f(x) = 0$ *and* $\lim_{x\to 2^+} f(x) = 0$

Since the limits from the left and right of 2 are equal, $\lim_{x\to 2} f(x) = 0$.

c)



$\lim_{x\to 1^-} f(x) = 0$ and $\lim_{x\to 1^+} f(x) = 1$

Since the limit from the left and right of 1 are not equal, $\lim_{x\to 1} f(x)$ does not exist.

3. a) A possible defining equation is $f(x) = \begin{cases} 2|x+1| - 2 & x < 0 \\ 3 & 0 \le x < 2 \\ -\sqrt{x-2} + 3 & x > 2 \end{cases}$

b) i) The limit as x approaches 0 does not exist since the limit from the left does not equal the limit from the right.

$\lim_{x\to 0^-} f(x) = 0$ $\lim_{x\to 0^+} f(x) = 3$ Therefore $\lim_{x\to 0} f(x)$ does not exist

ii) The limit as x approaches 2 equals 3 since the limit from the left equals the limit from the right.

$\lim_{x\to 2^-} f(x) = 3$ $\lim_{x\to 2^+} f(x) = 3$ Therefore $\lim_{x\to 2} f(x) = 3$

4. $\lim_{t\to 1^+} f(t) = 2 - b$ and $\lim_{t\to 1^-} f(t) = 3 + b$

If the limit as x approaches 1 does exist, the value of b could be calculated. Students may want to return to this question to find the value of b that makes the function continuous after outcome *L3* has been explored.

L3. Analyze the continuity of a function

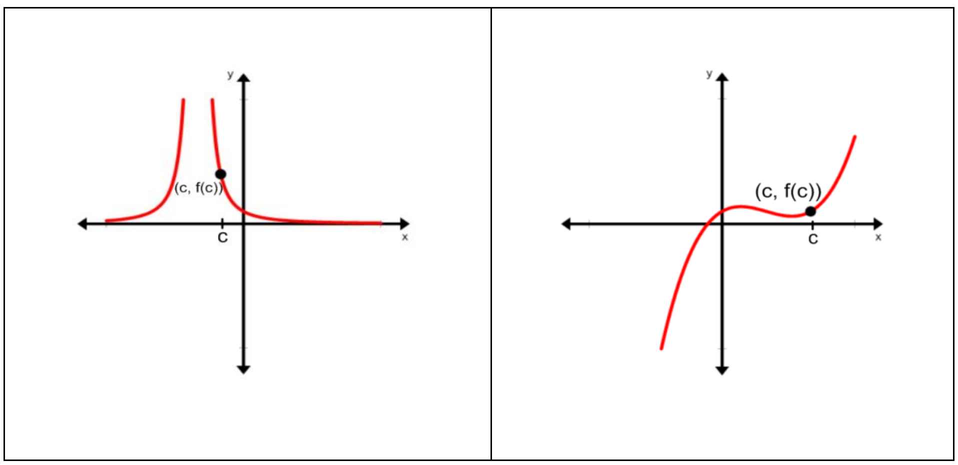
Achievement Indicators

- Demonstrate an understanding of the definition of continuity
- Determine if a function is continuous at a point, $x = a$.
- Determine if a function is continuous on a given open or closed interval.
- Explore properties of continuous functions
- Apply the Intermediate Value Theorem to continuous intervals of a function.

Continuity of a function

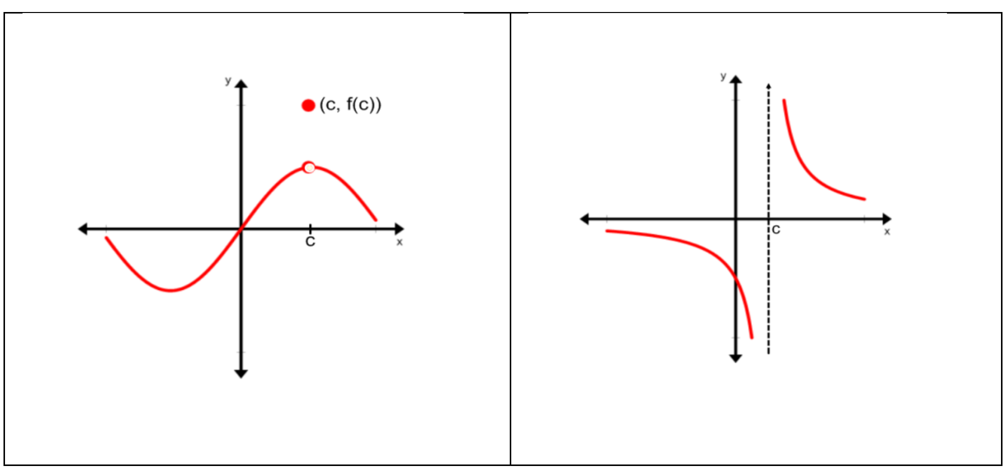
Continuity of a function at a given value can be determined by starting with any value of c in the domain of a function, and testing to see if it is possible to draw the graph of $f(x)$ at, and near the point $(c, f(c))$ without lifting a pencil from the paper. For example, in $Figure$ 3.1 below, each graph is a function $f(x)$ that is continuous at $x = c$.

Figure 3.1



In $Figure$ 3.2 the graphs illustrate functions that are not continuous at the point $x = c$. It is not possible to trace the graph around the point $(c, f(c))$ without lifting a pencil.

Figure 3.2



Definition of a Continuity

A function f is said to be continuous at $x = c$ if the following conditions are true:

I. $f(c)$ is defined

II. $\lim_{x \to c} f(x)$ exists

III. $\lim_{x \to c} f(x) = f(c)$

If condition III. is true, the first two conditions are implied.

Example 1

Show, by solving analytically, that the function $f(x) = \frac{\sqrt{x^2+2x-1}}{x-3}$ is continuous at the point where $x = 1$.

Solution

I. $f(1) = \frac{\sqrt{(1)^2 + 2(1) - 1}}{(1) - 3} = \frac{\sqrt{2}}{-2} = -\frac{\sqrt{2}}{2}$

II. $\lim_{x \to 1} \frac{\sqrt{x^2 + 2x - 1}}{x - 3} = \frac{\lim_{x \to 1} \sqrt{x^2 + 2x - 1}}{\lim_{x \to 1}(x - 3)} = \frac{\sqrt{\lim_{x \to 1}(x^2 + 2x - 1)}}{\lim_{x \to 1}(x - 3)} = \frac{\sqrt{(1)^2 + 2(1) - 1}}{(1) - 3} = -\frac{\sqrt{2}}{2}$

Therefore, $\lim_{x \to 1} f(x) = f(1)$, and f is continuous at $x = 1$.

Note: it is not necessary to check the right and left hand limits unless you are dealing with a point of discontinuity such as $x = 3$ in this example.

Example 2

For what values of x, if any, is the function $f(x) = \frac{x^2-4}{x^2+3x+2}$ discontinuous?

Solution

To solve questions for discontinuity, it is necessary to find the values in the domain of the function, f, that are undefined. Since $f(x)$ is a rational polynomial, these would be values in the domain which would cause the denominator to become 0, found by setting the denominator equal to 0 and solving for x.

$x^2 + 3x + 2 = 0$

$(x + 2)(x + 1) = 0$

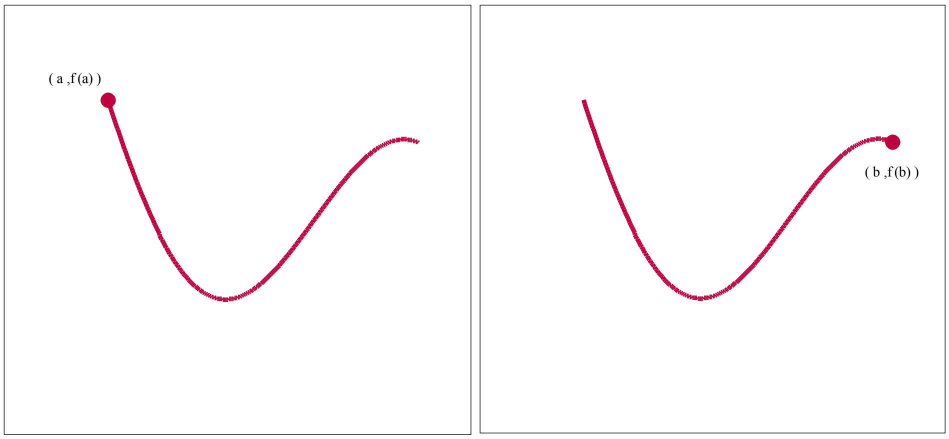
Therefore, either $x + 2 = 0$ or $x + 1 = 0$, and $x = -2$ or $x = -1$ would be the values in the domain where the function $f(x)$ would be discontinuous.

L3: Analyze the continuity of a function.

Continuity on an Interval

Continuity can also be defined from the left or the right of an endpoint as in the $Figure$ 3.3 at the points $(a, f(a))$ or $(b, f(b))$.

Figure 3.3



Continuity from the Left and Right

I. A function f is continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

II. A function f is continuous from the left at $x = b$ if $\lim_{x \to b^-} f(x) = f(b)$.

Continuity on a open and closed interval

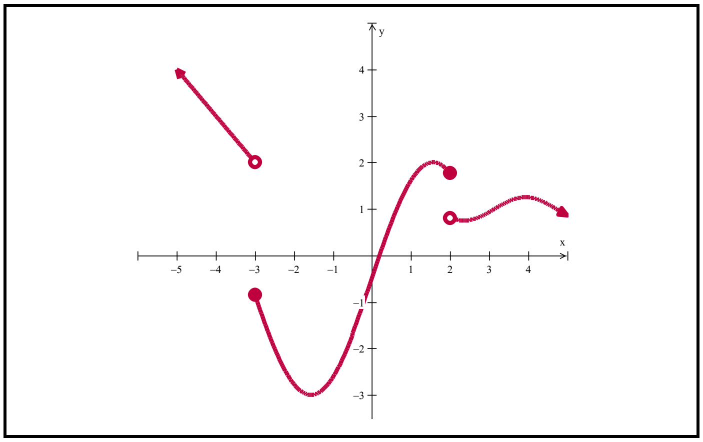
I. A function is continuous on an open interval (a, b) if it is continuous at every point c, where $c \in (a, b)$.

II. A function f is continuous on a closed interval $[a, b]$ if the following conditions are satisfied:
   - f is continuous on (a, b)
   - f is continuous from the right at a
   - f is continuous from the left at b

Example 3

The function f, shown in $Figure$ 3.4 below, is discontinuous at $x = -3$ and $x = 2$, but is continuous on each of the intervals $(-\infty, -3), [-3, 2], (2, +\infty)$.

Figure 3.4



Intermediate Value Theorem

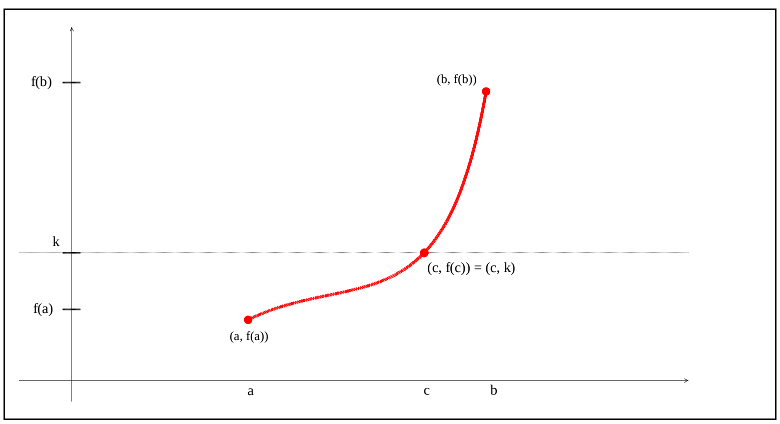
It is not necessary to prove the intermediate value theorem at this stage. However it can be introduced to students, using a simple geometric argument to support it.

Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$ inclusive, then there is at least one number c in the interval $[a, b]$ such that $f(c) = k$.

In $Figure$ 3.5 the function f is continuous on the interval $[a, b]$ and a number k that is between $f(a)$ and $f(b)$.

Figure 3.5

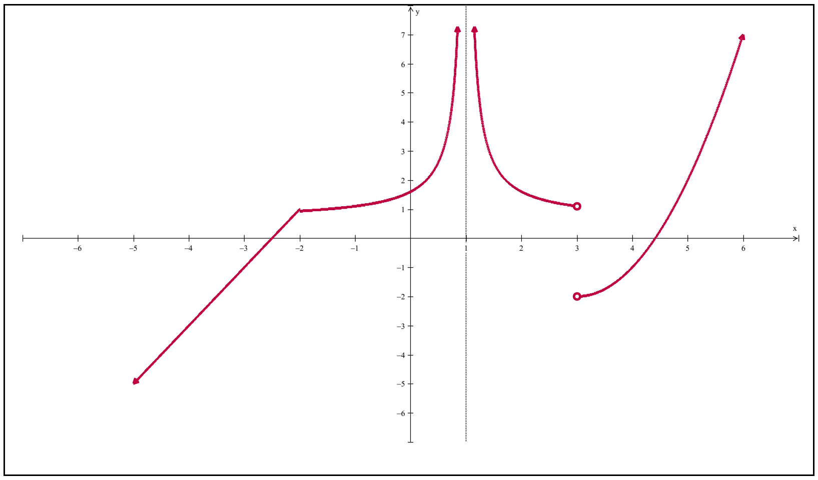


Because f is continuous on the interval, its graph can be drawn from $(a, f(a))$ to $(b, f(b))$ without lifting the pencil from the paper. As $Figure$ 3.5 indicates there is no way to do this unless the function crosses the horizontal line at $y = k$ at least once between $x = a$ and $x = b$. The coordinates of a point where this happens is either $(c, f(c))$ or (c, k).

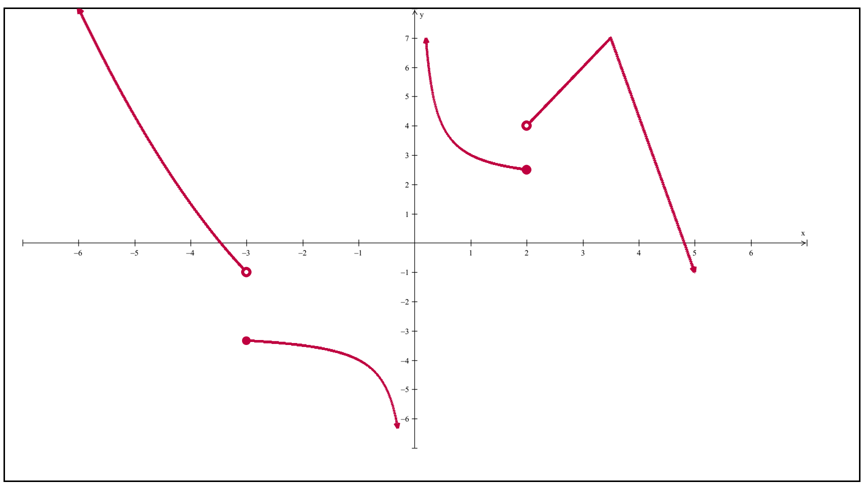
L3: Analyze the continuity of a function.

L3 Exercises

1. Given the graph of the function f below, find **all** the values in the domain of f at which f is **not** continuous.



2. Find **all** values in the function below where it is defined but **not** continuous.



L3: Analyze the continuity of a function.

In Exercises 3 to 8, use the definition of continuity and the properties of limits to show that the function is continuous at the given x value.

3. $f(x) = x^2 + 3x + 5$ at $x = 3$

4. $g(x) = x(x^2 - 3x + 5)$ at $x = 0$

5. $f(x) = \frac{x^2-4}{(x^2+4x+4)(x^2+2x+1)}$ at $x = 2$

6. $g(x) = \frac{x+3}{(x^2-x-1)(x^2+1)}$ at $x = -2$

7. $f(x) = \frac{x\sqrt{x}}{(x-4)^2}$ at $x = 16$

8. $g(x) = \frac{\sqrt{2-x^2}}{3x^2-1}$ at $x = -1$

Explain why the functions in questions 9 to12 are not continuous at the given number.

9. $f(x) = \frac{1}{(x-3)^3}$ at $x = 3$

10. $g(x) = \frac{(x^2+4)}{(x^2-x-2)}$ at $x = 2$

11. $h(x) = \frac{x^2+4x+3}{x^2-x-2}$ at $x = -1$

12. $k(x) = \begin{cases} \sin\frac{\pi}{x} & if\ x \neq 0 \\ 1 & if\ x \neq 0 \end{cases}$ at $x = 0$

13. Show that the function $f(x) = \frac{x^4-5x^2+4}{x-1}$ is not continuous on $[-3, 3]$ but does satisfy the conclusion of the Intermediate Value Theorem (that is, if k is a number between $f(-3)$ and $f(3)$, there is a number c between -3 and 3 such that $f(c) = k$).

Hint: What can be said about f on the intervals $[-3, -2]$ and $[2, 3]$?

14. For what value of b is the following function continuous at $x = 3$?

$$f(x) = \begin{cases} bx + 4 & if\ x \leq 3 \\ bx^2 - 2 & if\ x > 3 \end{cases}$$

Note: Question 4 in L2 can have an added question to find the value of b to make it continuous.+

L3: Analyze the continuity of a function.

L3 Solutions

1. $f(x)$ is not continuous at $x = 1$ and $x = 3$

2. $f(x)$ is defined but not continuous at $x = -3$ and $x = 2$. Note: $x = 0$ is not continuous but also is not defined.

3. $\lim_{x\to 3} x^2 + 3x + 5 = (3)^2 + 3(3) + 5 = 23$

   $f(3) = (3)^2 + 3(3) + 5 = 23$

   Since $f(3) = \lim_{x\to 3} f(x)$ then the function is continuous at $x = 3$.

4. $\lim_{x\to 0} x(x^2 - 3x + 5) = (0)((0)^2 - 3(0) + 5)0$

   $g(0) = (0)((0)^2 - 3(0) + 5) = 0$

   Since $g(0) = \lim_{x\to 0} g(x)$ then the function is continuous at $x = 0$.

5. $\lim_{x\to 2} \frac{x^2 - 4}{(x^2 + 4x + 4)(x^2 + 2x + 1)} = \frac{\lim_{x\to 2}(x^2 - 4)}{\lim_{x\to 2}(x^2 + 4x + 4)(x^2 + 2x + 1)} = \frac{(2)^2 - 4}{((2)^2 + 4(2) + 4)((2)^2 + 2(2) + 1)} = \frac{0}{144} = 0$

$f(2) = \frac{(2)^2 - 4}{((2)^2 + 4(2) + 4)\,((2)^2 + 2(2) + 1)} = \frac{0}{144} = 0$

Since $f(2) = \lim_{x\to 2} f(x)$ then the function is continuous at $x = 2$.

6. Since $g(-2) = \lim_{x\to -2} g(x) = \frac{1}{25}$ then the function is continuous at $x = -2$.

7. Since $f(16) = \lim_{x\to 16} f(x) = \frac{4}{9}$ then the function is continuous at $x = 16$.

8. Since $g(-1) = \lim_{x\to -1} g(x) = \frac{1}{2}$ then the function is continuous at $x = -1$.

9. $f(x)$is not continuous at $x = 3$ since the $f(3)$is undefined (denominator becomes 0 at $x = 3$).

10. $g(x)$is not continuous at $x = 2$ since the $g(2)$ is undefined (denominator becomes 0 at $x = 2$).

11. $h(x)$ is not continuous at $x = -1$ since the $h(-1)$ is undefined (denominator becomes 0 at $x = -1$).

12. $\lim_{x\to 0} k(x) \neq k(0)$, therefore not continuous.

13. Even though the function is not continuous at $x = 1$, $f(-3) = -10$ and $f(-2) = 0$, and the function is continuous on the interval $[-3, -2]$. As well, $f(2) = 0$ and $f(3) = 20$ and the function is continuous on the interval $[2, 3]$. Therefore, there would have to be a value of $k \in [f(-3), f(3)]$ that would satisfy the Intermediate value theorem.

14. $b = 1$

L4 Explore limits which involve infinity

Achievement Indicators

- Understand the definitions and the properties of limits at infinity.
- Use limits as $x \to \infty$ to find horizontal asymptotes of the function.
- Calculate limits of rational functions involving infinity
  $\lim_{x \to a} \frac{f(x)}{g(x)} = \pm\infty$, where $g(x) \neq 0$.
- Explore trigonometric graphs, such as sine and cosine, to show limits at infinity as an oscillating graph.

Vertical asymptotes

In evaluation of limits at infinity has been seen with reference to vertical asymptotes. When direct substitution of $x = c$ yields zero in the denominator and $x = c$ is not a removable point of discontinuity, the graph has a vertical asymptote at $= c$.

The only option on either side of the vertical asymptote is for the graph to increase without bound (to positive infinity) or decrease without bound (to negative infinity). The limit at $x = c$ does not exist.

If $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$, then there exists a vertical asymptote at $x = c$.

Once the vertical asymptote is identified, the behavior at, or at both sides of the vertical asymptote, can be examined.

Example 1

The function $f(x) = \frac{2}{x+5}$, has a vertical asymptote at $x = -5$.

As shown in **Table 3**, as x approaches -5 from the positive direction (from the right), $f(x)$ gets (infinitely) bigger and bigger. As x approaches -5 from the negative direction (from the left), $f(x)$ gets (infinitely) smaller and smaller.

Table 3

$$\lim_{x \to -5^+} \frac{2}{x+5} = \infty \qquad \lim_{x \to -5^-} \frac{2}{x+5} = -\infty$$

x	$f(x)$	x	$f(x)$
-4.9	20	-5.1	-20
-4.99	200	-5.01	-200
-4.999	2000	-5.001	-2000
-4.9999	$20\,000$	-5.0001	$-20\,000$

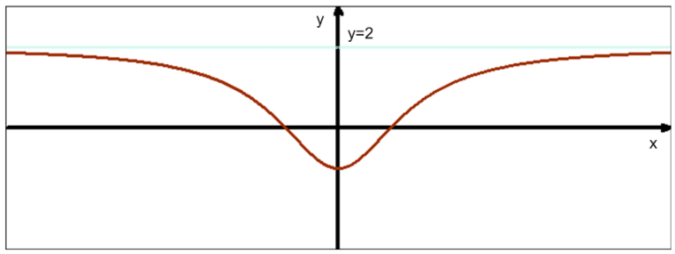
Limits at Infinity

Example 1

Investigate the behavior of the rational function $f(x) = \frac{2x^2-2}{x^2+2}$

Table 4

x	$f(x)$
0	-1
± 1	0
± 2	1
± 3	1.454545
± 5	1.77777
± 10	1.941176
± 100	1.999400
± 1000	1.9999940



As x grows larger and larger the values of $f(x)$ get closer and closer to 2.

Evaluating the behaviour of the function as x grows infinitely large in either the positive or negative direction, is also referred to as evaluating the limits at infinity as x approaches $\pm\infty$. This is an analyses of the end behavior of the function is as either $x \to -\infty$ or $x \to \infty$. The end behavior of a graph is very important. Limits at infinity must be one-sided;

$$\lim_{x\to\infty} f(x) \qquad or \qquad \lim_{x\to-\infty} f(x)$$

Some functions do not have end behaviour. For example, $f(x) = \sqrt{x}$ does not have end behavior at $-\infty$, because the domain of the graph $x \geq 0$, *so* x cannot approach $-\infty$.

If a graph does have end behavior there are only a few things that can happen.

- The graph can increase without bound.
  For example:
  $$f(x) = x^2; \qquad \lim_{x\to\infty} x^2 = \infty \qquad \text{or} \qquad \lim_{x\to-\infty} x^2 = \infty$$
- The graph can decrease without bound.
  For example:
  $$f(x) = -x^2; \qquad \lim_{x\to\infty} -x^2 = -\infty \qquad \text{or} \qquad \lim_{x\to-\infty} -x^2 = -\infty$$
- The graph can oscillate between two fixed values.
  For example:
  $$f(x) = \sin x\,, then \ \lim_{x\to\infty} f(x) = DNE$$
- The graph can taper off toward a specific finite y-value, L.
  For example:
  If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ then there exists a **horizontal asymptote** at $y = L$

When taking limits at infinity, we're essentially looking for the existence or non-existence of any horizontal asymptotes.

Horizontal asymptotes are **not** discontinuities. A horizontal asymptote describes the end behavior of the graph. A graph can cross a horizontal asymptote any number of times as the limit only exists when x increases or decreases without bound. Horizontal asymptotes are a special characteristic of many rational functions (a polynomial over a polynomial, $y = \frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$) and can be determined by examining the growth of the numerator in relation to the denominator.

There are **3 Cases** based on the comparison of the degrees of the leading terms of the numerator and denominator. The leading terms are not necessarily the ones written in front, but rather the terms with the largest power of x. The main technique for these is to divide each term in the numerator and denominator by the highest power of x in the denominator.

Note: Careful consideration must be given to the forms $\frac{\pm\infty}{\pm\infty}$ and $\infty - \infty$ during computations of limits of infinity. Often it is incorrectly concluded that $\frac{\pm\infty}{\pm\infty} = 1$, or that $\infty - \infty = 0$. These are examples of indeterminant forms which can usually be algebraically manipulated to obtain a solution.

Case 1

The degree of denominator is greater than the degree of the numerator.

In this case, for extremely large values of x, the denominator is getting larger faster than the numerator, so the fractions are getting smaller and approaching zero. All terms other than the term with the highest power are relatively small for very large x's, thus they do not significantly contribute to the growth. In this case, there will be a horizontal asymptote at $y = 0$.

$$\lim_{x\to -\infty} \frac{4x^3 - 6x^2 + 8}{7x^5 + 6x^2 - 9x + 2} = \lim_{x\to -\infty} \frac{\frac{4x^3}{x^5} - \frac{6x^2}{x^5} + \frac{8}{x^5}}{\frac{7x^5}{x^5} + \frac{6x^2}{x^5} - \frac{9x}{x^5} + \frac{2}{x^5}} = \lim_{x\to -\infty} \frac{\frac{4}{x^2} - 0 + 0}{7 + 0 - 0 + 0} = \frac{0}{7} = 0$$

Between step 2 and 3 of the above example we are simplifying the limit process as we know that those terms will approach 0. It doesn't matter how much larger the denominator's degree is than the numerator's, only that it is. The greater the degree difference, the faster the values go to zero.

Case 2

The degree of the numerator and denominator are exactly the same.

In this case, the top and bottom are essentially growing at the same rate. Again the trailing terms are insignificantly small compared to the leading terms. This means that the fractions or ratios are approaching the ratio of the leading terms, or the leading terms' coefficients. In this case, there will always be an horizontal asymptote at $y = \frac{leading\ coefficient}{leading\ coefficient}$.

Example1:

$$\lim_{x\to\infty} \frac{12x^4 - 6x^3 + 5}{5x^4 - x^2 + x + 2} = \lim_{x\to\infty} \frac{\frac{12x^4}{x^4} - \frac{6x^3}{x^4} + \frac{5}{x^4}}{\frac{5x^4}{x^4} - \frac{x^2}{x^4} + \frac{x}{x^4} + \frac{2}{x^4}} = \lim_{x\to\infty} \frac{12 - \frac{6}{x} + \frac{5}{x^4}}{5 - \frac{1}{x^2} + \frac{1}{x^3} + \frac{2}{x^4}} = \lim_{x\to\infty} \frac{12 - 0 + 0}{5 - 0 + 0 + 0} = \frac{12}{5}$$

Horizontal Asymptote at $y = \frac{12}{5}$

Example 2:

$$\lim_{x\to-\infty}\frac{10x^2+7x}{3x^2-5x+1}=\lim_{x\to-\infty}\frac{\frac{10x^2}{x^2}+\frac{7x}{x^2}}{\frac{3x^2}{x^2}-\frac{5x}{x^2}+\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{10+\frac{7}{x}}{3-\frac{5}{x}+\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{10+0}{3-0+0}=\frac{10}{3}$$

Horizontal Asymptote at $y=\frac{10}{3}$

Case 3

The degree of the numerator is greater than the degree of the denominator.

In this case, the numerator's is growing much faster than the denominator either in the positive or negative directions (increasing or decreasing without bound). The limit will not exist and there will be no horizontal asymptote. In Summary, if the numerator has higher degree then the denominator, then we will always have the function approaching $+\infty$ or $-\infty$ as the end behavior of our rational function.

$$\lim_{x\to\infty}\frac{3-4x^2}{5x-1}=\lim_{x\to\infty}\frac{\frac{3}{x}-\frac{4x^2}{x}}{\frac{5x}{x}-\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{3}{x}-4x}{5-\frac{1}{x}}=\lim_{x\to\infty}\frac{0-4x}{5-0}=\frac{-4x}{5}$$

The limit approaches $-\infty$.

To determine what **type** of infinity (positive or negative), any representative huge x-value can be substituted into the leading terms to determine the sign, in the direction that is being approached.

$$\lim_{x\to\infty}\frac{-5x^6-8x+14}{8x^3+3x^2+2x-7} \qquad and \qquad \lim_{x\to-\infty}\frac{-2x^7-4x^4+10}{-6x^3-x^2+x+5}$$

$$=\lim_{x\to\infty}\frac{-5(\text{positive})^6-\text{does not matter}}{8(\text{positive})^3+\text{does not matter}} \qquad =\lim_{x\to-\infty}\frac{-2(\text{negative})^7-\text{does not matter}}{-6(\text{negative})^3-\text{does not matter}}$$

$$=\lim_{x\to\infty}\frac{-5(\text{positive})}{8(\text{positive})} \qquad =\lim_{x\to-\infty}\frac{-2(\text{negative})}{-6(\text{negative})}$$

$$=-\infty \qquad =\infty$$

Sometimes, the degrees of the numerator and denominator are not as explicit. This often occurs when either the numerator or denominator are under a radical. For these types of problems, we can use a similar analysis.

Compute:

$$\lim_{x\to\infty}\frac{\sqrt{2x^3-x}}{3x^2-6} \qquad <----\,this\ is\ an\ indeterminate\ form\ of\ type\ \frac{\infty}{\infty}$$

$$=\lim_{x\to\infty}\frac{\sqrt{2x^3-x}}{3x^2-6}\left(\frac{\sqrt{\frac{1}{x^4}}}{\frac{1}{x^2}}\right) \qquad Note:\sqrt{\frac{1}{x^4}}=\frac{1}{x^2}$$

$$=\lim_{x\to\infty}\frac{\sqrt{\frac{2x^3}{x^4}-\frac{x}{x^4}}}{\frac{3x^2}{x^2}-\frac{6}{x^2}}=\lim_{x\to\infty}\frac{\sqrt{\frac{2}{x^3}-\frac{1}{x^3}}}{3-\frac{6}{x^2}}=\lim_{x\to\infty}\frac{\sqrt{0-0}}{3-0}=0$$

Students should be aware at this point that when evaluating the end behavior of any function in their toolkit, they are actually finding the limit of that function as x approaches infinity.

L4: Explore limits which involve infinity.

Examples:

(students should verify their conclusions by sketching the graphs for each function)

$\lim_{x\to +\infty} 3^x = \infty \quad rationale; \; 3^\infty = \infty$

$\lim_{x\to -\infty} \left(\frac{2}{7}\right)^x = \infty \quad rationale; \; \left(\frac{2}{7}\right)^{-\infty} = \left(\frac{7}{2}\right)^{\infty} = \infty$

$\lim_{x\to +\infty} \ln(x) = \infty \quad rationale; \quad As\ x\ increases\ in\ value, \ln(x)\ increases\ in\ value.$

$\lim_{x\to +\infty} \sqrt{x-2} = \infty \quad rationale; \quad As\ x\ increase\ in\ value, \sqrt{x-2}\ increases\ in\ value.$

$\lim_{x\to -\infty} |x| = \infty \quad rationale; \; As\ x\ becomes\ larger\ negatively, |x|\ becomes\ larger\ positively.$

$\lim_{x\to +\infty} (x-4)^5(2x-1)^2(3x+4) = \infty \quad rationale; \quad (large\ +)^5(large\ +)^2(large\ +) = +\infty$

$\lim_{x\to -\infty} (x+3)^3(x+1)^2 = -\infty \quad rationale; \quad (large\ -)^3(large\ -)^2 = -\infty$

$\lim_{x\to +\infty} \frac{1}{x-4} = 0 \quad rationale; \quad As\ x\ increases\ in\ value, \frac{1}{x-4}\ becomes\ very\ small.$

$\lim_{x\to +\infty} \frac{1}{x^2-2x-3} = 0 \quad rationale; \quad As\ x\ increases\ in\ value, \frac{1}{x^2-2x-3}\ becomes\ very\ small.$

L4 Exercises

1. Explain in your own words the meaning of each of the following:

a) $\lim\limits_{x\to -\infty} f(x) = 6$ b) $\lim\limits_{x\to \infty} f(x) = -9$ c) $\lim\limits_{x\to 4^+} f(x) = \infty$ d) $\lim\limits_{x\to 6^-} f(x) = -\infty$

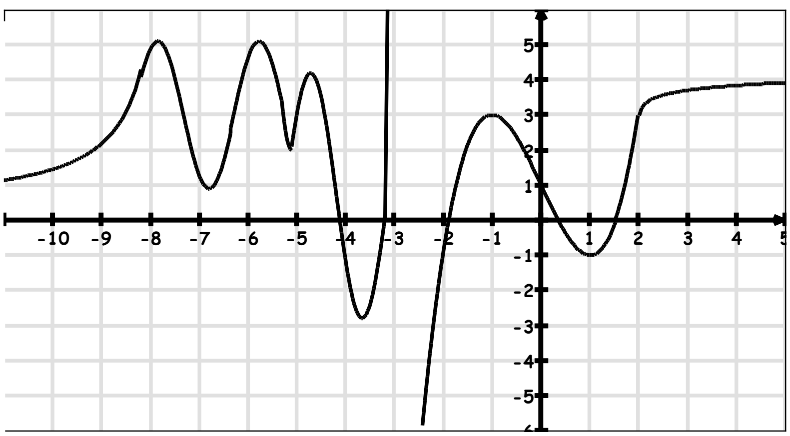
2. Can the graph of $y = f(x)$ intersect :
   a) a vertical asymptote?
   b) a horizontal asymptote?

   Explain your conclusions.

3. For the function whose graph is shown below, determine the following:

   a) $\lim\limits_{x\to\infty} f(x)$ b) $\lim\limits_{x\to -\infty} f(x)$ c) $\lim\limits_{x\to -3^-} f(x)$ d) $\lim\limits_{x\to -3^+} f(x)$

   e) State the equations of the vertical and horizontal asymptotes.



4. Evaluate $\lim\limits_{x\to\infty} \dfrac{x^2}{2^x}$ by using a table of values.

5. Evaluate the following limits, if possible.

a) $\lim\limits_{x\to\infty} \dfrac{5x^4 - 7x^3 + 7x^2 - 1}{3x^4 + 2x^3}$ b) $\lim\limits_{x\to -\infty} \dfrac{x^5 - x^2}{x^3 - 2x}$ c) $\lim\limits_{x\to\infty} \dfrac{9x^2 - x + 8}{2x^4 + x^3 - 7}$

d) $\lim\limits_{x\to -\infty} \dfrac{12x^2 - 6x^3 + 5x^4 + 9x^5}{3x^5 + 2x^4 - 4x^3 + 2x}$ e) $\lim\limits_{x\to\infty} \dfrac{5x^3 - 4x^2 - 5x}{4x^3 + 3x}$ f) $\lim\limits_{x\to\infty} \left(\dfrac{1}{8}\right)^x + \dfrac{x^3 - 4x^2 - 5x}{4x^3 + 3x} - 7$

g) $\lim\limits_{x\to\infty} \left(\dfrac{7}{3}\right)^{-x} + \dfrac{4x^2 - 5x}{2x^2 + 1} - 9$ h) $\lim\limits_{x\to -\infty} \left(\dfrac{1}{5}\right)^{2x}$ i) $\lim\limits_{x\to\infty} \dfrac{(-2x^2 - 3)(x + 1)}{2 - 5x^3}$

j) $\lim\limits_{x\to -\infty} 12 - \left(\dfrac{6}{5}\right)^x$ k) $\lim\limits_{x\to\infty} \dfrac{10x^2}{\sqrt{4x^4 + 1}}$ l) $\lim\limits_{x\to\infty} \dfrac{2x}{\sqrt{x^2 - 2} - 3x}$

L4: Explore limits which involve infinity.

m) $\lim_{x\to\infty} \frac{4-\frac{3}{x}}{5x^2+1}$

n) $\lim_{x\to\infty} \frac{6x}{2x-1} - \frac{x+5}{3x-4}$

o) $\lim_{x\to\infty} x - \sqrt{x^2+1}$

p) $\lim_{x\to-\infty} \frac{|x-8|}{x-8}$

q) $\lim_{x\to\infty} 2x - \sqrt{4x^2+6x}$

r) $\lim_{x\to-\infty} (x-3)^2(x+1)^5$

s) $\lim_{x\to-\infty} (2x-3)^3(x+1)^3$

6. Determine the horizontal and vertical asymptotes and any other points of discontinuity of the functions.

a) $f(x) = \frac{x^2+8x-20}{2x^2+x-6}$

b) $f(x) = \frac{x^2-3x-40}{x^3+8}$

c) $f(x) = \frac{x^4-2x^3-63x^2}{x^2-10x+16}$

d) $f(x) = \frac{10x^3-18x}{x^3-x^2-2x}$

e) $f(x) = \frac{9-x^2}{16-2x^3}$

f) $f(x) = \frac{x-7}{\sqrt{3x^2+10x+8}}$

7. State the vertical asymptotes. Describe the behavior of $f(x)$ on each side of the vertical asymptote(s):

a) $f(x) = \frac{2}{x^2-9}$

b) $f(x) = \frac{x^2+x}{x+5}$

c) $f(x) = \tan x$

L4 Solutions

1. a) As x gets increasing larger in the negative direction, the value of the function approaches 6. The constant value of $y = 6$ will not change.

   b) As x gets increasing larger in the positive direction the value of the function approaches -9. The constant value of $y = -9$ will not change

   c) As x approaches 4 from the right hand side, the value of the function gets increasingly larger, approaching positive infinity.

   d) As x approaches 6 from the left hand side, the value of the function becomes more negative, approaching negative infinity

2. a) No, the function is undefined at a vertical asymptote.

   b) Yes, a horizontal asymptote occurs for very large negative or positive values of x. Therefore, the graph may cross the asymptote when x is small in relation to $\pm\infty$.

3. a) 4
   b) 1
   c) ∞
   d) $-\infty$
   e) $HA: \ y = 1 \ and \ y = 4$
   $VA: \ x = \ -3$

4. $\lim\limits_{x\to\infty} \dfrac{x^2}{2^x} = 0$

5.
a) $\frac{5}{3}$ b) ∞ c) 0 d) 3 e) $\frac{5}{4}$ f) $-\frac{27}{4}$ g) -7 h) ∞ i) $\frac{2}{5}$ j) 12

k) 5 l) -1 m) 0 n) $\frac{8}{3}$ o) 0 p) -1 q) $-\frac{3}{2}$ r) $-\infty$ s) ∞

6. a) $HA\!: \ y = 1/2 \quad VA: x = 3/2, x = -2$
   b) $HA\!: \ y = 0 \quad VA: x = -2,$
   c) $HA\!: \ y = none \quad VA: x = 2, x = 8$
   d) $HA\!: \ y = 10 \quad VA: x = -1, \quad x = 2, \quad Point\ of\ Discontinuity\ (0, 9)$
   e) $HA\!: \ y = 1/2 \quad VA: x = \ 2$
   f) $HA\!: \ y = \dfrac{1}{\sqrt{3}} \ VA: \ x = -2, x = \dfrac{-4}{3}.\ The\ graph\ is\ undefined\ between\ the\ VA.$

7. a) $VA\!: x = -3; \ x = 3$
   $Behaviour\!: \ \lim\limits_{x\to -3^-} \dfrac{2}{x^2 - 9} = +\infty \qquad \lim\limits_{x\to -3^+} \dfrac{2}{x^2 - 9} = -\infty \qquad \lim\limits_{x\to 3^-} \dfrac{2}{x^2 - 9} - \infty \qquad \lim\limits_{x\to 3^+} \dfrac{2}{x^2 - 9} = \infty$

   b) $VA\!: x = -5$
   $Behaviour\!: \ \lim\limits_{x\to -5^-} \dfrac{x^2 + x}{x + 5} = -\infty \qquad \lim\limits_{x\to -5^+} \dfrac{x^2 + x}{x + 5} = +\infty$

   c) $VA\!: x = \frac{n\pi}{2},\ where\ n\ is\ an\ odd\ integer.$